Combinatorial modulus on boundaries of right-angled hyperbolic buildings

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Abstract

The *Loewner property* is a quasi-conformal property that can be used to prove Mostow like rigidity of some hyperbolic spaces. Here we present the *combinatorial Loewner property (CLP)* and boundaries of hyperbolic spaces that satisfy the CLP. This property is a weak version (but conjecturally equivalent) of the Loewner property.

Boundary and rigidity of hyperbolic spaces

Let X be a hyperbolic space and ∂X its boundary equipped with a visual metric. Following ideas of Mostow, Gromov, Pansu, Bourdon-Pajot we aim to prove the implications bellow. Let $f : X \to X$ be a quasi-isometry, then:

$$(\partial f \text{ is a quasi-conformal homeo.}) \underset{\text{Loewner}}{\Longrightarrow} (\partial f \text{ is a conformal homeo.})$$
$$\underset{\text{Liouville}}{\Longrightarrow} (\text{there exists an isometry } F : X \to X \text{ with } \partial F = \partial f.)$$

Boundaries of hyperbolic spaces that satisfy the Loewner property are rare because they require the knowledge of the *conformal dimension*.

Combinatorial modulus of curves

Approximation G_k of Z a compact metric space. For $k \in \mathbb{N}$, let G_k be *minimal and homogeneous* covering of Z by *quasi-balls* of diameter 2^{-k}.
G_k-combinatorial p-modulus. Let $\mathcal{F} \neq \emptyset$ be a set of curves of Z. A function $\rho : G_k \to \mathbb{R}_+$ is *$\mathcal{F}$-admissible* if:

$$L_\rho(\gamma) = \Sigma_{b \cap \gamma \neq \emptyset}\, \rho(b) \geq 1 \text{ for any } \gamma \in \mathcal{F}.$$

For $p \geq 1$ the *G_k-combinatorial p-modulus* of $\mathcal{F}$ is:

$$\text{Mod}_p(\mathcal{F}, G_k) = \inf_\rho \{\sum_b \rho(b)^p\},$$

where the infimum is taken over all the $\mathcal{F}$-admissible functions.

Prop.: $\text{Mod}_p(\cdot, G_k)$ is a discrete outer measure on the curves of Z *i.e*

1. if $\mathcal{F}_1 \subset \mathcal{F}_2$ then $\text{Mod}_p(\mathcal{F}_1, G_k) \leq \text{Mod}_p(\mathcal{F}_2, G_k)$,
2. $\text{Mod}_p(\cup_{i=1,\dots,m} \mathcal{F}_i, G_k) \leq \Sigma_{i=1,\dots,m} \text{Mod}_p(\mathcal{F}_i, G_k)$.

The CLP

Idea. ∂X verifies the CLP if *the amount of curves joining two continua is controlled by the relative distance between them.*

The CLP is easier to prove than the Loewner property: it does not require the knowledge of the conformal dimension and it can be proved geometrically.

Examples

- $\partial \mathbb{H}^n$ for $n \geq 3$ and $\partial \Delta$ for Δ a right-angled Fuchsian building (cf. [2]) satisfy the Loewner property.
- The Sierpiński carpet embedded in $\mathbb{E}^2$ satisfies the CLP (cf. [1]). It is conjectured that it is Q.C to a Loewner spaces.
- Spaces with local cut points do not satisfy the CLP.

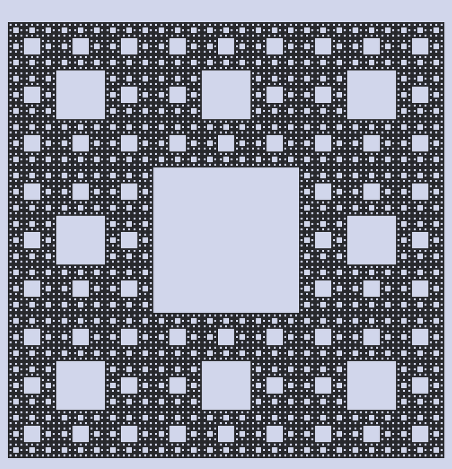


Figure 1: Sierpiński carpet

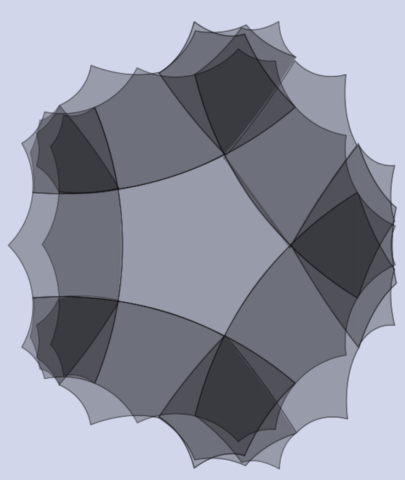


Figure 2: Neighbourhood of a chamber in Δ

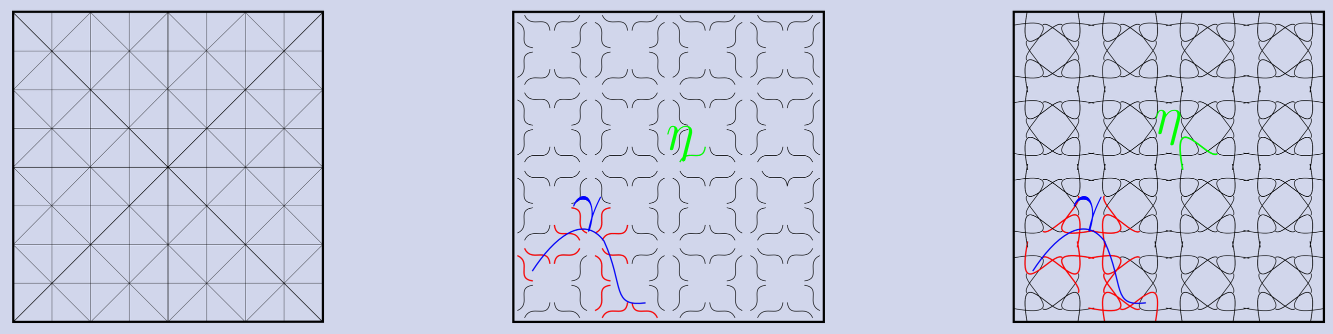
A geometric criterion for the CLP

To build a wire netting.

(S) : *For all $\epsilon > 0$, there exists F a finite set of bi-Lipschitz homeo., s.t for any curve η large enough and any curve $\gamma \subset Z$, the set $\cup_{f \in F} f(\eta)$ contains a sub-curve η' with $d_{C^0}(\gamma, \eta') < \epsilon$.*

According to [1]: **(S)** $\Longrightarrow$ CLP.

On boundaries of Coxeter groups we may verify **(S)** thanks to reflections.



A right-angled hyperbolic building

The graph product Γ. Let D be the regular RA dodecahedron in $\mathbb{H}^3$ and $q \in \mathbb{N}$, $q \geq 3$. Let $\{\sigma_1, \dots, \sigma_{12}\}$ be the faces of D. We define

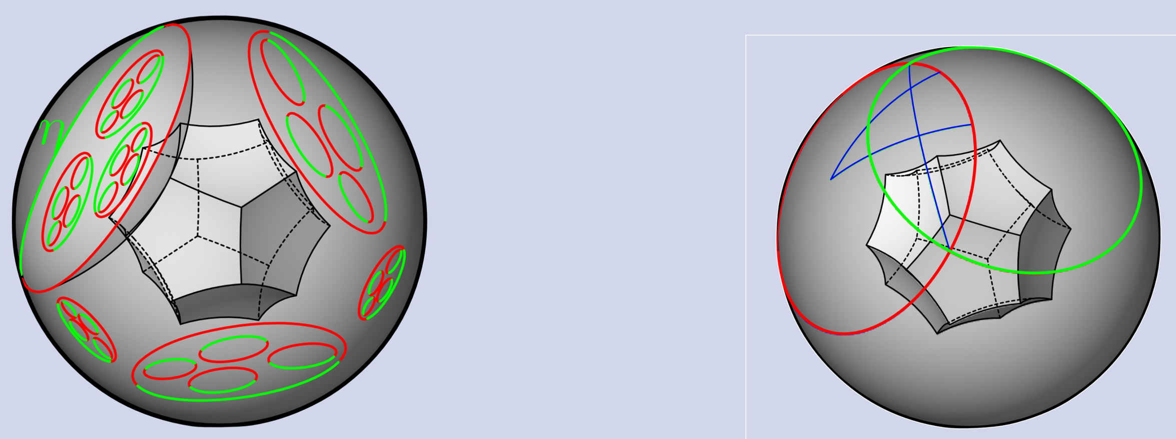
$$\Gamma = \langle s_1, \dots, s_{12} | s_i^q = 1 \text{ et } [s_i, s_j] = 1 \text{ if } \sigma_i \perp \sigma_j \text{ in } D \rangle,$$
$$\text{and } W = \langle s_1, \dots, s_{12} | s_i^2 = 1 \text{ et } [s_i, s_j] = 1 \text{ if } \sigma_i \perp \sigma_j \text{ in } D \rangle.$$

The group Γ acts geometrically on Σ a *right- angled hyperbolic building of type W* and we identify $\partial \Sigma \simeq \partial \Gamma$. The Coxeter group W is the reflection group generated by the faces of D in $\mathbb{H}^3$.

Parabolic Limit Sets

In ∂W and in $\partial \Gamma$, *parabolic limit sets* are obstructions to prove **(S)**.

In the Coxeter group. If M is a wall in W and $\eta \subset \partial M$, **(S)** is not verified with $F \subset W$. However **(S)** is verified using the symmetries of D.



CLP on $\partial\Gamma$

Thanks to **(S)** in ∂W and to a control of $\text{Mod}_p(\cdot, G_k)$ in $\partial \Gamma$ by a combinatorial modulus computed in ∂W, we obtain

Theorem [3]

The boundary $\partial \Gamma$ equipped with a visual metric satisfies the CLP.

References

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