

# NOTES ON BOUNDARIES OF CAT(0) GROUPS AND SPACES AFTER KIM RUANE

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ABSTRACT. CAT(0) spaces are non-positively curved metric spaces. In this series of lectures, we discuss how concepts such as: the geometry of the space, the topology on the boundary, the metric on the boundary, the action of a group, the algebraic structure of the group are related. We will insist on explicit and intuitive examples and give references for the proofs.

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These notes have been made after the Lectures gave by Pr. Kim Ruane during the GEAR Junior retreat at the University of Michigan, Ann Arbor, on May 25 and 26, 2014. The results that are not mentioned with a precise reference can be found with a proof in [BH99].

## TERMINOLOGY AND NOTATION

Along these notes we will use the following conventions. Let  $(X, d)$  be a metric space, a *geodesic* ray (resp. line) is an isometry from  $([0, +\infty), |\cdot - \cdot|)$  (resp.  $(\mathbb{R}, |\cdot - \cdot|)$ ) to  $(X, d)$ . The Euclidean and hyperbolic spaces of dimension  $d$  are respectively  $\mathbb{E}^d$  and  $\mathbb{H}^d$ . A metric space  $X$  is called a *proper* metric space if closed balls are compact subsets. Moreover  $X$  is

called *geodesic* if for any pair of points  $x, y \in X$  there exists a geodesic segment between them. In the following, we will say that a group  $G$  has *virtually* a property  $(\mathcal{P})$  if there exists a finite index subgroup  $H$  of  $G$  that satisfy  $(\mathcal{P})$ .

**Definition 1.** *If  $X$  is a metric space, we say that the group  $G$  acts geometrically on  $X$  if*

- $G < \text{Isom}(X)$ ,
- $G$  acts properly discontinuously,
- the quotient  $X/G$  is compact.

Along these notes  $(X, d)$  is a proper geodesic and complete metric space.

## 1. FIRST LECTURE

**1.1. To describe the curvature with the shape of triangles.** Originally the curvature is a notion of differential geometry. Here we present different ways to characterize non-positively curved space only in metric terms. A good reference for the basic facts presented above is [BH99].

*CAT(0) spaces and groups.* For any triple of points  $x, y, z \in X$ , let  $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$  be such that

$$d(x, y) = d_{\mathbb{E}}(\bar{x}, \bar{y}), \quad d(y, z) = d_{\mathbb{E}}(\bar{y}, \bar{z}), \quad \text{and} \quad d(z, x) = d_{\mathbb{E}}(\bar{z}, \bar{x}).$$

The Euclidean triangle  $\bar{x}, \bar{y}, \bar{z}$  is called the *triangle of comparison* of  $x, y, z \in X$ .

**Definition 2.** *We say that  $(X, d)$  is a CAT(0) space if for any  $x, y, z \in X$  and any  $p$  belonging to some geodesic segment between  $x$  and  $y$  then*

$$d(z, p) \leq d_{\mathbb{E}}(\bar{z}, \bar{p}).$$

Where  $\bar{p}$  is the unique point in  $\mathbb{E}^2$  such that  $d(x, p) = d_{\mathbb{E}}(\bar{x}, \bar{p})$  and  $d(y, p) = d_{\mathbb{E}}(\bar{y}, \bar{p})$ .

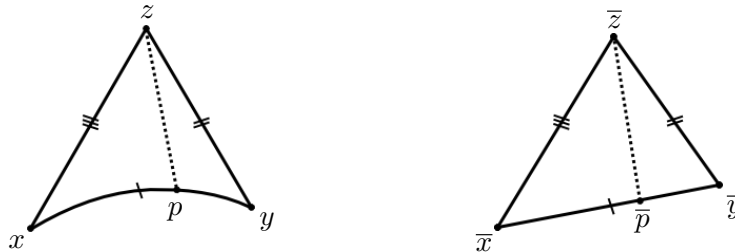


FIGURE 1. Triangle in  $X$  and comparison triangle in  $\mathbb{E}^2$

With common words, this definition means that *triangles in  $X$  are, at least, as thin as triangles in  $\mathbb{E}^2$ .*

From the definition it comes that if  $X$  is a CAT(0) space then:

- (1)  $X$  is contractible,
- (2) for any  $x, y \in X$  there exists a unique geodesic segment containing  $x$  and  $y$ ,
- (3) the dimension of the isometrically embedded Euclidean spaces is bounded. The embedded Euclidean spaces in a CAT(0) space are called *flats*.

**Remark 1.** The acronym CAT stands for Cartant-Alexandrov-Topogonov has been introduced by M. Gromov. CAT(0) spaces are also called Hadamard spaces in the literature.

**Definition 3.** A group  $G$  is called a CAT(0) group if there exists a CAT(0) space on which  $G$  acts geometrically.

Note that, as a consequence of the definition of CAT(0) spaces, the Cayley graph of a CAT(0) group  $G$  is a CAT(0) space if and only if  $G$  is a free group with the canonical presentation. From the definition it comes that if  $G$  is a CAT(0) group then

- (1)  $G$  is finitely presented,
- (2)  $G$  satisfies a quadratic isoperimetrical inequality,
- (3) there is a bound on the rank of the free abelian subgroups.

**Example 1.** In the following examples,  $X$  is a CAT(0) metric space that admits a geometric action by  $G$ .

- $G = \mathbb{Z}^d$ ,  $X = \mathbb{E}^d$ .
- $G = \pi_1(\Sigma_g)$ , where  $\Sigma_g$  is a hyperbolic surface of genus  $g \geq 2$ ,  $X = \mathbb{H}^2$ .
- More generally if  $X$  is the universal cover of a non-positively curved Riemannian manifold and  $G$  is its fundamental group. Then  $X$  and  $G$  are CAT(0).
- $G = F_2$  the free group of rank 2,  $X = T_4$  the infinite tree of constant valence 4.

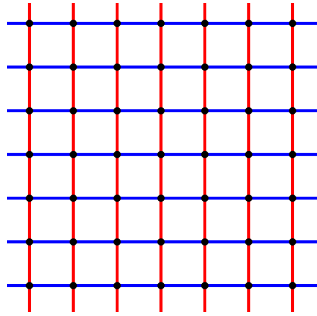


FIGURE 2. Cayley graph of  $\mathbb{Z}^2$

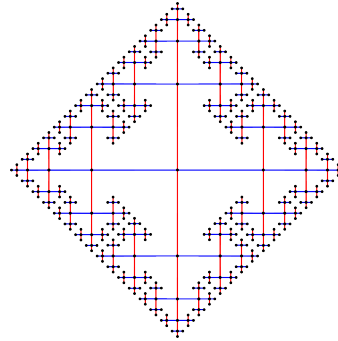


FIGURE 3. Cayley graph of  $F_2$

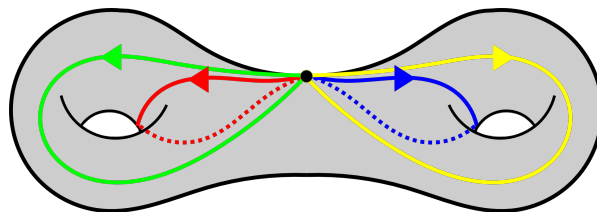


FIGURE 4. A system of generators of  $\pi_1(\Sigma_2)$

A nice property of the CAT(0) condition is that it is preserved by the product. Note that the metric considered on the product, is the sum of the metrics on the factors. Naturally this property extends to CAT(0) groups. For instance  $X = T_4 \times \mathbb{R}$  and  $G = F_2 \times \mathbb{Z}$  are CAT(0).

**Remark 2.** We can define a locally CAT(0) condition. If  $X$  is locally CAT(0) and simply connected then it is globally CAT(0).

The main object of these lectures is to study boundaries of CAT(0) spaces. Yet, many questions and ideas in the CAT(0) world are inspired by the theory of  $\delta$ -hyperbolic spaces. This theory gives another way to characterize the non-positive curvature in metric terms.

*$\delta$ -hyperbolic spaces.* The notion of  $\delta$ -hyperbolicity is due to M. Gromov. Good references about hyperbolic spaces and groups are [GdlH90] and again [BH99].

**Definition 4.** We say that  $X$  is a  $\delta$ -hyperbolic space (or hyperbolic in the sense of Gromov) if there exists a constant  $\delta > 0$  such that for every geodesic triangle  $[x, y] \cup [y, z] \cup [z, x] \subset X$  and every  $p \in [x, y]$ , the following inequality is verified

$$\text{dist}(p, [y, z] \cup [z, x]) \leq \delta.$$

A finitely generated group that acts geometrically on a  $\delta$ -hyperbolic space  $X$  is called a hyperbolic group.

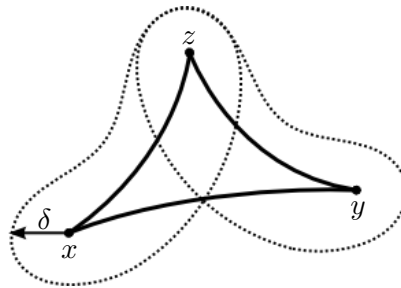


FIGURE 5

With common words a metric space is  $\delta$ -hyperbolic if *large triangles of  $X$  are thinner than triangles of  $\mathbb{E}^2$* . Naturally a hyperbolic space (resp. group) does not contain any embedded Euclidean plane (resp.  $\mathbb{Z}^2$  as a subgroup).

A major feature of the  $\delta$ -hyperbolicity condition is that it is preserved by quasi-isometries. In particular if a finitely generated group  $G$  acts geometrically on a metric space  $X$  then  $G$  equipped with a word metric is quasi-isometric to  $X$ . As a consequence:

**Theorem 1.** Let  $G$  be a hyperbolic group and  $\mathcal{G}$  be a Cayley graph of  $G$ . Then  $\mathcal{G}$  is a hyperbolic space.

Finite groups and virtually cyclic groups are hyperbolic groups. We call them *elementary* hyperbolic groups. *Non-elementary* examples of hyperbolic groups are given in the following.

**Example 2.** In the following,  $X$  is a hyperbolic space that admits a geometric action by  $G$ .

- $G = \pi_1(\Sigma_g)$ , where  $\Sigma_g$  is a compact hyperbolic surface of genus  $g$ ,  $X = \mathbb{H}^2$ .
- $X$  is the universal cover of a compact negatively curved Riemannian manifold and  $G$  is its fundamental group.
- $G = F_2$  the free group of rank 2,  $X = T_4$  the infinite tree of constant valence 4.

The  $\delta$ -hyperbolicity condition is not preserved by the direct product. Indeed if  $G, G'$  are infinite  $\delta$ -hyperbolic groups then  $G \times G'$  contains a copy of  $\mathbb{Z}^2$ .

*CAT(-1) spaces.* These spaces are a third way to compare the triangles. Here we substitute the comparison triangles in  $\mathbb{E}^2$  with comparison triangles in  $\mathbb{H}^2$ .

For any triple of points  $x, y, z \in X$ , let  $\hat{x}, \hat{y}, \hat{z} \in \mathbb{H}^2$  be such that

$$d(x, y) = d_{\mathbb{H}}(\hat{x}, \hat{y}), \quad d(y, z) = d_{\mathbb{H}}(\hat{y}, \hat{z}), \quad \text{and} \quad d(z, x) = d_{\mathbb{H}}(\hat{z}, \hat{x}).$$

**Definition 5.** We say that  $(X, d)$  is a CAT(-1) space if for any  $x, y, z \in X$  and any  $p$  belonging to some geodesic segment between  $x$  and  $y$  then

$$d(z, p) \leq d_{\mathbb{H}}(\hat{z}, \hat{p}).$$

Where  $\hat{p}$  is the unique point in  $\mathbb{H}^2$  such that  $d(x, p) = d_{\mathbb{H}}(\hat{x}, \hat{p})$  and  $d(y, p) = d_{\mathbb{H}}(\hat{y}, \hat{p})$ . A group  $G$  is called a CAT(-1) group if there exists a CAT(-1) space on which  $G$  acts geometrically.

**Example 3.** The hyperbolic spaces and groups presented in Example 2 are CAT(-1).

*Relations between these definitions.* We insist on the fact that the  $\delta$ -hyperbolicity condition is a condition that holds at large scale whereas CAT(0) and CAT(-1) conditions hold at any scale. In particular, any compact metric space or finite group is hyperbolic.

It is direct to see that CAT(-1) spaces (resp. groups) are CAT(0) and  $\delta$ -hyperbolic. The converse lead to a first open question:

*Question 1.* Let  $G$  be a non-elementary hyperbolic group. Is  $G$  CAT(0)?

As soon as a CAT(0) space  $X$  (resp. group  $G$ ) contains  $\mathbb{R}^2$  as an isometric embedding (resp.  $\mathbb{Z}^2$  as a subgroup) then  $X$  (resp.  $G$ ) is not  $\delta$ -hyperbolic. To find  $\mathbb{Z}^2$  in a group  $G$  is a common way to prove that  $G$  is not hyperbolic. In the CAT(0) world we can say more.

**Theorem 2** (Flat Plane Theorem (see p.459 [BH99])). *Let  $G$  be a group acting geometrically on a CAT(0) space  $X$ . Then  $G$  is  $\delta$ -hyperbolic if and only if  $X$  does not contain an isometrically embedded copy of  $\mathbb{E}^2$ .*

This theorem is a partial result to the following open question:

*Question 2.* Is it true that a CAT(0) group  $G$  is  $\delta$ -hyperbolic if and only if it does not contain  $\mathbb{Z}^2$  as a subgroup?

**1.2. Boundaries of CAT(0) spaces.** In this subsection,  $(X, d)$  is a CAT(0) space. In these notes, we present different topologies and metrics on the boundaries of CAT(0) spaces.

**Definition 6.** Let  $\gamma, \gamma' : [0, +\infty) \rightarrow X$  be two geodesic rays. We say that  $\gamma$  and  $\gamma'$  are asymptotic if

there exists  $K > 0$  such that  $d(\gamma(t), \gamma'(t)) \leq K$  for any  $t \in [0, +\infty)$ .

In this case we write  $\gamma \sim \gamma'$  and we define

$$\partial X := \{\gamma \text{ geodesic ray in } X\} / \sim .$$

We call  $\partial X$  the boundary of  $X$ .

In the following we will explore the different topologies that we can use on  $\partial X$ . As this definition of the boundary is a definition at large scale, it holds also for  $\delta$ -hyperbolic spaces.

*Visual topology on  $\partial X$ .* Using the completeness of  $X$ , we can prove the following property.

**Proposition 1.** For any  $x \in X$ , for any  $\gamma \in \partial X$  there exists a unique geodesic ray  $\gamma'$  such that  $\gamma'(0) = x$  and  $\gamma \sim \gamma'$ .

Now we fix a base point  $x_0 \in X$  and we set

$$\partial_{x_0} X := \{\gamma : \gamma \text{ is geodesic ray in } X \text{ with } \gamma(0) = x_0\}.$$

Let  $y_0 \in X$ , because of the previous proposition,  $\partial_{x_0} X$  is canonically in bijection with  $\partial X$  and with  $\partial_{y_0} X$ .

Note that in a  $\delta$ -hyperbolic space the proposition may not be true. Yet, up to a bounded distance, the identification holds *i.e*

$$\partial X = \{\gamma : \gamma \text{ is geodesic ray in } X \text{ with } \gamma(0) = x_0\} / \sim .$$

**Definition 7.** Let  $\gamma \in \partial_{x_0} X$  and  $\epsilon, R > 0$  with  $R \gg 1$  and  $\epsilon \ll 1$ . We set

$$N(\gamma, \epsilon, R) = \{\gamma' : d(\gamma(R), \gamma'(R)) < \epsilon\}.$$

Let  $\mathcal{T}_{x_0}$  be the topology generated by the basis of neighborhood  $\{N(\gamma, \epsilon, R) | \gamma \in \partial_{x_0} X, \epsilon > 0, R > 0\}$ . This topology is called a visual topology on  $\partial X$  and we use the notation  $\partial_{x_0} X$  for the topological space  $(X, \mathcal{T}_{x_0})$ .

We can check that the space  $\partial_{x_0} X$  is a compact space and that this topology is invariant up to changing the base point.

**Theorem 3.** Let  $x_0, y_0 \in X$  then  $\partial_{x_0} X$  is homeomorphic to  $\partial_{y_0} X$  and the homeomorphism is given by the canonical bijection.

Hence the visual topology on  $\partial X$  is unique and from now we use the notation  $\partial_\infty X$  for the boundary equipped with this topology.

**Remark 3.** • We can also equip,  $\partial X$  with a topology that comes from a definition of  $\partial X$  as an inverse limit. It turns out that this inverse limit topology is equivalent to the visual one.

- Note that we can define a topology on  $\bar{X} = X \cup \partial X$  coherent with the visual topology. Then  $\bar{X}$  is a compact set and  $X$  is a dense subset of  $\bar{X}$ .

**Example 4.**

- $X = \mathbb{E}^d, \partial_\infty X = \mathbb{S}^{d-1}$ .
- $X = \mathbb{H}^d, \partial_\infty X = \mathbb{S}^{d-1}$ .
- $X = T_4, \partial_\infty X = \mathcal{C}$  where  $\mathcal{C}$  is the Cantor set.

**Remark 4.** As we saw in the previous examples, some spaces with very different geometry may have the same visual boundary. This lead to the following general question:

*Question 3.* What can be said about the geometry of  $X$  knowing  $\partial_\infty X$ ? Is it possible to distinguish these spaces thanks to a group action on the boundary?

If a CAT(0) space is a direct product of CAT(0) spaces, then the boundary is decomposable as a topological join.

**Definition 8.** Let  $A$  and  $B$  be two compact topological spaces. The join of  $A$  and  $B$  is

$$A * B := (A \times [0, 1] \times B) / R$$

where  $R$  is the equivalence relation generated by

- $(a, 0, b_1) \sim (a, 0, b_2)$  for any  $a \in A$  and  $b_1, b_2 \in B$ ,
- $(a_1, 1, b) \sim (a_2, 1, b)$  for any  $a_1, a_2 \in A$  and  $b \in B$ .

If  $A$  is made of two points we write  $A * B = \Sigma B$  and we call  $\Sigma B$  the suspension of  $B$ .

Intuitively, to get  $A * B$  you take the disjoint union of  $A$  and  $B$  and between any couple of points  $a \in A$  and  $b \in B$  you attach a segment.

**Proposition 2.** Let  $X$  and  $Y$  be two CAT(0) spaces. Then  $\partial_\infty(X \times Y) = \partial_\infty X * \partial_\infty Y$ . In particular  $\partial_\infty(X \times \mathbb{R}) = \Sigma \partial_\infty X$ .

**Example 5.**  $X = T_4 \times \mathbb{R}$  is CAT(0) and  $\partial_\infty X = \Sigma \mathcal{C}$  where  $\mathcal{C}$  is the Cantor set.

**Example 6.** Let  $C = \{(x_1, x_2) \in \mathbb{E}^2 | x \geq 0 \text{ and } y \geq 0\}$ . Let  $C_0, \dots, C_4$  be five copies of  $C$  and let  $X = (C_0 \sqcup \dots \sqcup C_4) / \sim$  where  $\sim$  is the equivalence relation obtained by attaching cyclically the horizontal boundary of  $C_i$  to the vertical boundary of  $C_{i+1}$  for any  $i \in \mathbb{Z}/5\mathbb{Z}$ . We call  $X$  the fake plane.

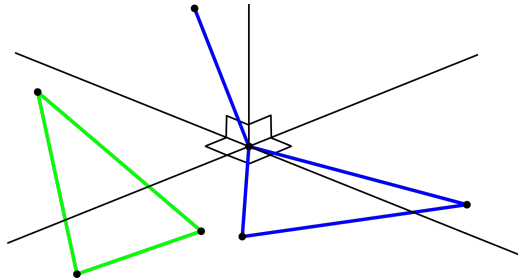


FIGURE 6. Triangles in the fake plane

Note that, as we observe on the Figure 6, the geodesic between two points  $p$  and  $q$  respectively contained in  $C_0$  and  $C_2$  passes through  $(0, 0)$ .

The fake plane  $X$  is CAT(0) and  $\partial_\infty X = \mathbb{S}^2$ . It is somehow a pathological example of CAT(0) space that we will try to eliminate thanks to group actions.

**1.3. Isometries of CAT(0) spaces.** Here we discuss how a geometric action of a group  $G$  on a CAT(0) space  $X$  extends to the boundary and how it can help to understand the geometry of the space.

In this subsection,  $G$  is a group acting geometrically on the CAT(0) space  $X$ .

*Classification of the isometries.* A classification of the isometries by their fixed points is well known in hyperbolic geometry. Such a classification also apply in the CAT(0) world.

**Definition 9.** Let  $g \in \text{Isom}(X)$ , we set

$$|g| := \inf\{d(x, gx) | x \in X\} \text{ and } \text{Min}(g) := \{x \in X | d(x, gx) = |g|\}.$$

We call  $|g|$  the translation length of  $g$ . If  $\text{Min}(g) \neq \emptyset$  then  $g$  is called a semi-simple element.

Notice that for any  $g, h \in G$ , one has  $|hgh^{-1}| = |g|$  and  $\text{Min}(hgh^{-1}) = h.\text{Min}(g)$ . In particular, if  $g$  and  $h$  commute then  $\text{Min}(g)$  is stabilized by  $h$ . Thanks to the completeness of  $X$  we can prove the following property.

**Proposition 3.** If  $G$  acts geometrically on a CAT(0) space  $X$  then any  $g \in G$  is semi-simple.

**Definition 10.** Let  $g \in \text{Isom}(X)$ .

- (1) If  $|g| = 0$  then  $g$  is of finite order and is called an elliptic element. In this case  $\text{Min}(g) = \text{Fix}(g)$ .
- (2) If  $|g| > 0$  then  $g$  is of infinite order and is called a hyperbolic element.

When  $g \in G$  is hyperbolic, the structure of  $\text{Min}(g)$  is very rich. That is what we describe here. In the following proposition, we call *Euclidean strip* in  $X$  a subset of  $X$  isometric to  $\{(x, y) \in \mathbb{E}^2 | 0 \leq x \leq t\}$  for some  $t > 0$ .

**Proposition 4.** Let  $g$  be a hyperbolic isometry.

- There exists a geodesic line  $\ell : \mathbb{R} \rightarrow X$  such that  $g$  acts by translation on  $\ell$ . We say that  $\ell$  is an axis of  $g$ .
- Let  $\ell, \ell'$  be two distinct axis of  $g$ . Then there exists a Euclidean strip in  $X$  bounded by  $\ell$  and  $\ell'$ .

**Definition 11.** Let  $g \in \text{Isom}(X)$  and  $\ell$  be an axis of  $g$ . Pick a point  $x_0 \in \ell$ . Then we define

$$Y := \pi_\ell^{-1}(x_0) \cap \text{Min}(g)$$

where  $\pi_\ell$  is the projection map of  $X$  over  $\ell$ .

**Proposition 5.**  $\text{Min}(g) = Y \times \mathbb{R}$ . Moreover  $Y$  and  $\text{Min}(g)$  are closed and convex sets.



- Example 7.** (1) Let  $g$  be a hyperbolic isometry of  $\mathbb{H}^2$ . Then  $\text{Min}(g) = \{*\} \times \mathbb{R}$ .  
 (2) Let  $G = \mathbb{Z}^2$ . For any  $g \in G$ ,  $\text{Min}(g) = \mathbb{E}^2$ .  
 (3) Let  $G = \pi_1(K)$ , where  $K$  is the Klein bottle. Then  $G = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ . The element  $b$  acts by translation on  $\mathbb{E}^2$  hence  $\text{Min}(b) = \mathbb{E}^2$ . The element  $a$  acts by a translation followed by a reflection, hence  $\text{Min}(a) = \{*\} \times \mathbb{R}$ . But  $a^2$  acts by translation and  $\text{Min}(a^2) = \mathbb{E}^2$ .

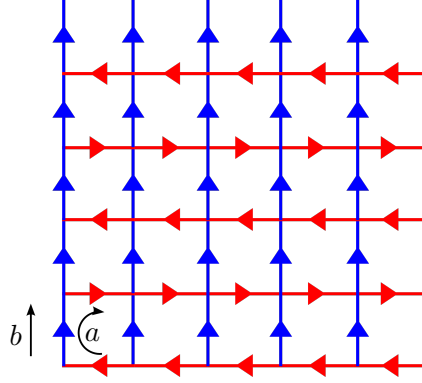


FIGURE 7. Universal cover of the Klein bottle

**Theorem 4.** Let  $C(g)$  be the centralizer of a hyperbolic element  $g \in G$

- (1)  $C(g)$  stabilizes  $\text{Min}(g)$ .
- (2)  $C(g)$  acts geometrically on  $\text{Min}(g)$ .
- (3)  $C(g)/\langle g \rangle$  acts geometrically on  $Y$ .

*Extension of isometries on the boundary.* It is now natural to extend the action of the group on the boundary. This leads to the following general question:

*Question 4.* How the boundary can provide informations about the algebra in the group?

Let  $g \in \text{Isom}(X)$ . Because  $g$  maps a geodesic to a geodesic, it is straightforward to see that  $g$  extends into a homeomorphism on the boundary  $\bar{g} : \partial_\infty X \rightarrow \partial_\infty X$ . Moreover  $g$  acts by translation on  $\text{Min}(g)$  hence it is straightforward to see that  $\partial_\infty \text{Min}(g) \subset \text{Fix}(\bar{g})$ . But actually we have more.

**Theorem 5** (see Theorem 3.3 in [Rua01]). *If  $g$  is hyperbolic then  $\partial_\infty \text{Min}(g) = \text{Fix}(\bar{g})$ .*

Moreover,  $\partial_\infty \text{Min}(g)$  is an attractive set for the action of  $\langle g \rangle$  on  $\partial_\infty X$ .

**Example 8.** In the following we consider the geometric action of  $G$  on a CAT(0) space  $X$ .

- (1) Let  $G = \mathbb{Z}^2$  and  $X = \mathbb{E}^2$ . Then for every  $g \in G$  one has  $\bar{g} = \text{id}_{\mathbb{S}^1}$
- (2) Let  $G = \pi_1(K)$  and  $X = \mathbb{E}^2$ . With the notation of Example 7 (3), one has  $\bar{b} = \text{id}_{\mathbb{S}^1}$ ,  $\bar{a} = R_{\mathbb{S}^1}$ , and  $\bar{a}^2 = \text{id}_{\mathbb{S}^1}$ . Where  $R$  is a reflection in  $\mathbb{E}^2$  relative to an axis containing the origin and  $\mathbb{S}^1$  is considered as the unit circle in  $\mathbb{E}^2$ .

- (3)  $G = \pi_1(\Sigma_g)$  acts on  $\mathbb{H}^2$ . If  $g \neq \text{id}_{\mathbb{H}^2}$  then  $\bar{g} \neq \text{id}_{\mathbb{S}^1}$ . More precisely  $\bar{g}$  is of infinite order with a North-South dynamic.

Thanks to these examples we can see that the actions on the boundaries are very different even if the topologies are the same. We can sometime distinguish the geometry of the space thanks to the boundary.

Quasi-isometries are the natural maps between hyperbolic spaces. Such a map extends as a homeomorphism between the boundaries. As a consequence, if  $G$  is a group acting geometrically on two hyperbolic spaces  $X$  and  $Y$ , then there exists a  $G$ -equivariant homeomorphism between  $\partial_\infty X$  and  $\partial_\infty Y$ . Hence it is natural to ask the general question:

*Question 5.* How does a quasi-isometry between CAT(0) spaces extends to the boundaries?

A partial answer to this question is provided by C. Hruska and B. Kleiner:

**Theorem 6** ([HK05]). *If  $X$  has isolated flats. Then  $\partial X$  is invariant under the extension of quasi-isometry maps.*

Yet, note that sometimes the boundary of a CAT(0) group is not uniquely defined *i.e* there exists a CAT(0) group  $G$  acting geometrically on two CAT(0) metric spaces  $X$  and  $Y$  such that the boundaries  $\partial_\infty X$  and  $\partial_\infty Y$  are not homeomorphic.

**Example 9.** The first example of such a group have been provided by C. Croke and B. Kleiner in [CK00]. The group in question is the right-angled Artin group  $A_\Gamma$  where  $\Gamma$  is the graph given by the Figure 8.



FIGURE 8. The graph  $\Gamma$

This group admits the following presentation

$$A_\Gamma = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle$$

and acts geometrically on the universal cover  $X_\alpha$  of the complex  $S_\alpha$  obtained by gluing tree torus along closed curves with the identification of the Figure 9. Then for  $\beta \neq \alpha$  it comes

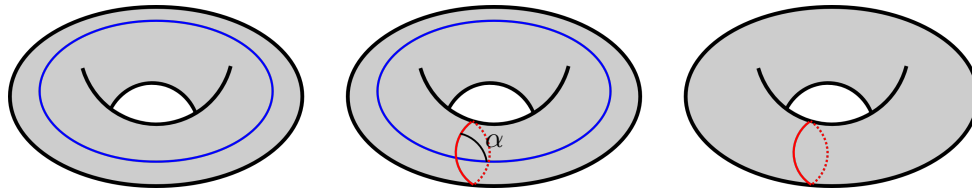


FIGURE 9. The complex  $S_\alpha$

that  $\partial_\infty X_\alpha$  and  $\partial_\infty X_\beta$  are not homeomorphic.

Eventually, the following theorem is an example of how the algebraic structure of the group may be related to the boundary.

**Theorem 7** (see Theorem 3.4 in [Rua01]). *Let  $G$  be a group acting geometrically on a CAT(0) space  $X$ . For  $g \in G$ ,  $\bar{g} = \text{id}$  on  $\partial_\infty X$  if and only if  $g$  is virtually central in  $G$ .*

## 2. SECOND LECTURE

**2.1. Non-trivial examples of non-CAT(0) groups.** The following theorem provides non-trivial examples of non-CAT(0) groups.

**Theorem 8** (Flat Torus Theorem). *If  $G$  acts geometrically on a CAT(0) space  $X$  and if  $G$  contains a subgroup  $A \simeq \mathbb{Z}^n$ , then there exists an isometric embedding  $F : \mathbb{E}^n \rightarrow X$  such that the image is stabilized by  $A$ . In this case,  $A$  has a torus action on  $F(\mathbb{E}^n)$ .*

**Example 10** (Gersten's example of non-CAT(0) group (see [Ger94])). Let  $G = F_3 \rtimes_\varphi \mathbb{Z}$  be the group given by the following presentation

$$G = \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2 \rangle.$$

We can write  $G = \langle a, b, c, t \mid tat^{-1} = a, b^{-1}tb = at, c^{-1}tc = a^2t \rangle$ . As  $a$  and  $t$  commutes,  $\langle a, t \rangle \simeq \mathbb{Z}^2 < G$ .

Now assume that  $G$  acts on a CAT(0) space. According to the flat torus theorem,  $\langle a, t \rangle$  stabilize a Euclidean plane in  $X$  with a torus action. This means that  $a$  and  $t$  act by translation on  $\mathbb{E}^2$ . Yet as we saw previously, for any  $g, h \in G$  one has  $|hgh^{-1}| = |g|$ . Hence we obtain  $|t| = |at| = |a^2t|$ . Which lead to a contradiction.

As a consequence of the flat torus theorem and of this non-example, any group that contains the previous group  $G$  as a subgroup is not CAT(0). In particular, S.M Gersten proved that  $Out(F_n)$  for  $n \geq 3$  and  $Aut(F_3)$  are not CAT(0). Moreover if  $\Sigma_g$  is a surface of genus  $g \geq 3$ , then  $MCG(\Sigma_g)$  is not CAT(0) (see [BH99, Chap. II-7.26]).

**Theorem 9.** *If  $G$  is a CAT(0) group acting on  $X$  and  $\mathbb{Z}^n$  is central in  $G$ , then  $G$  is virtually a direct product  $H \times \mathbb{Z}^n$ .*

With the previous theorem and the Theorem 4 comes the following corollary:

**Corollary 1.** *For each hyperbolic element  $g \in G$  one has  $C(g) \cong H \times \langle g \rangle$ .*

**2.2. The Tits metric on  $\partial X$ .** In this subsection,  $(X, d)$  is a CAT(0) space. Here we introduce an alternative topology to the visual one, the Tits topology. This topology is in fact induced by a metric on  $\partial X$ .

Let  $\gamma, \gamma'$  be two geodesic rays such that  $\gamma(0) = \gamma'(0) = x_0 \in X$ . For any  $t \in (0, +\infty)$  let  $\overline{\angle_{x_0}(\gamma(t), \gamma'(t))}$  be the angle at  $x_0$  of the Euclidean triangle of comparison  $\overline{x_0}, \gamma(t), \gamma'(t)$ . The *angle of comparison* based on  $x_0 \in X$  is

$$\angle_{x_0}(\gamma, \gamma') = \lim_{t \rightarrow 0} \overline{\angle_{x_0}(\gamma(t), \gamma'(t))}.$$

**Definition 12.** *For  $\gamma, \gamma' \in \partial X$  we set*

$$\angle(\gamma, \gamma') = \sup_{x_0 \in X} \{\angle_{x_0}(\gamma_{x_0}, \gamma'_{x_0})\}.$$

Where  $\gamma_{x_0}$  and  $\gamma'_{x_0}$  are the representatives of  $\gamma, \gamma'$  based on  $x_0 \in X$ .

It is an exercise to check that  $\angle(\cdot, \cdot)$  is a metric defined on  $\partial X$  and bounded by  $\pi$ . We call  $\angle(\cdot, \cdot)$  the *angular metric* on  $\partial X$ .

**Example 11.** (1) Let  $X$  be a CAT(0) and  $\delta$ -hyperbolic metric space. In a  $\delta$ -hyperbolic space the following property holds.

*For any  $\gamma, \gamma' \in \partial X$  there exists a geodesic line  $\ell : (-\infty, +\infty) \rightarrow X$  such that  $\gamma = \lim_{t \rightarrow -\infty} \ell(t)$  and  $\gamma' = \lim_{t \rightarrow +\infty} \ell(t)$ .*

As a consequence for any  $\gamma, \gamma' \in \partial X$  one has  $\angle(\gamma, \gamma') = \pi$  and equipped with the topology induced by this metric,  $\partial X$  is discrete.

- (2) Let  $X = \mathbb{E}^2$  then  $(\partial X, \angle)$  is isometric to a circle of length  $2\pi$ .  
(3) Let  $X = (C_0 \sqcup \dots \sqcup C_4) / \sim$  be the fake plane (we use the notation of the Example 6). Let  $\gamma, \gamma'$  be two geodesic rays based on  $x_0$ . Assume, for instance, that  $\gamma \subset C_0$  and  $\gamma' \subset C_2$ . Then  $\angle(\gamma, \gamma') = \pi$ .

The third example above motivates the following definition to eliminate this kind of situation.

We call *path* in the metric space  $(\partial X, \angle)$  a continuous map  $p : [0, 1] \rightarrow \partial X$ . The *length*  $L(p)$  of the path  $p$  is defined as follow

$$L(p) = \sup_{0=t_0 < t_1 < \dots < t_n=1} \sum_{i=0}^{n-1} \angle(p(t_i), p(t_{i+1})),$$

where the supremum is taken over all possible partitions of  $[0, 1]$  and  $n$  is unbounded. Then a path  $p$  is called a *rectifiable path* if  $L(p) < \infty$ .

**Definition 13.** Let  $\gamma, \gamma' \in \partial X$ . We set

$$d_T(\gamma, \gamma') = \inf\{\ell(p) \mid p \text{ is a rectifiable path from } \gamma \text{ to } \gamma'\}.$$

If there is no rectifiable path between  $\gamma$  and  $\gamma'$  then we set  $d_T(\gamma, \gamma') = \infty$ . The metric  $d_T(\cdot, \cdot)$  on  $\partial X$  is called the *Tits metric*. The topological space obtained induced by this metric is called the *Tits boundary* and is denoted  $\partial_T X$ .

- Example 12.** (1) Let  $X$  be a  $\delta$ -hyperbolic space. For any  $\gamma, \gamma' \in \partial X$  distinct, one has  $d_T(\gamma, \gamma') = \infty$ .  
(2) Let  $X$  be the fake plane, then  $\partial_T X$  is a circle of length  $5\pi/2$ .  
(3) Let  $X$  and  $Y$  be two CAT(0) spaces then  $\partial_T(X \times Y) \simeq \partial_T X * \partial_T Y$ .  
(4) Let  $X = \mathbb{H}^2 \times \mathbb{R}$ . On the one hand  $\partial_\infty X = \Sigma \mathbb{S}^1 \simeq \mathbb{S}^2$  is of topological dimension 2. On the other hand  $\partial_T X$  is the suspension of an uncountable discrete set and is of topological dimension 1.

The fourth example above illustrates the fact that  $\text{id}_{\partial X} : \partial_T X \rightarrow \partial_\infty X$  is a continuous bijection but is not a homeomorphism. In particular the following theorem holds.

**Theorem 10** (Kleiner). *Let  $X$  be a CAT(0) space admitting a geometric action from a group  $G$ . Let  $F$  be the dimension of the largest flats of  $X$ . Then  $\dim_{\text{Topo}}(\partial_T X) = F - 1$ .*

**2.3. Examples of non-hyperbolic CAT(0) groups.** Let  $\varphi \in \text{Out}(F_2) = \text{GL}_2(\mathbb{Z})$  and let  $G = F_2 \rtimes_{\varphi} \mathbb{Z}$ . If  $\varphi$  is of finite order, it is straightforward to see that  $G$  is virtually  $F_2 \times \mathbb{Z}$ .

- (1) If  $\varphi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  then  $G = \pi_1(\mathbb{S}^3 \setminus K)$  where  $K$  is the eight knot. We know that  $\mathbb{S}^3 \setminus K$  is a hyperbolic 3-manifold with cusps. Hence  $G$  acts properly by isometry but not cocompactly on  $\mathbb{H}^3$ . Let  $X$  be  $\mathbb{H}^3$  with removed open horoballs such that the quotient  $M := X/G$  is compact.

We equip  $X$  with the metric  $d$  defined as follows:

- on the interior of  $X$  the metric  $d$  coincide with the hyperbolic metric,
- on the horospheres  $d$  coincide with the Euclidean metric.

The consequence of a removed open horoball in  $\mathbb{H}^3$  is a removed disc on the boundary  $\mathbb{S}^2$ .

**Theorem 11** ([Rua05] Cor. 4.2). *The metric space  $(X, d)$  is CAT(0),  $\partial_{\infty} X$  is a Sierpiński carpet and  $\partial_T X$  is a disjoint collection of points and circles. The points are the limit of the hyperbolic part. The circles are the limit of the horoballs.*

- (2) If  $\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  then  $G = \langle a, b, t \mid tat^{-1} = a, tat^{-1} = ab \rangle$ . Up to a change of presentation  $G = \langle x, y, z \mid [x, y] = 1, zxz^{-1} = y \rangle$ . This means that  $G$  is a HNN extension  $G = (\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}}$ .

**2.4. Final examples.** If  $G$  is the fundamental group of two torus glued along a closed curve, then

$$G = (\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}).$$

With the notation  $G = \langle a, b \rangle *_{\langle b \rangle} \langle b, c \rangle$  we can prove that the sequence  $\{c^n\}_{n \geq 1}$  accumulates to a point on the boundary and that is a point of non-local connectivity. To each  $c^n$  correspond a flat that is a copy of  $\langle a, b \rangle$ . If  $G$  is the fundamental group of the space obtained by gluing a torus to  $\Sigma_g$  along a curve (with  $g \geq 2$ ) *i.e*

$$G = \pi_1(\Sigma_g) *_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}).$$

then  $\partial_{\infty} G$  is made of a disjoint collection of points and circles.

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